

# On Discontinuous First Order Implicit Boundary Value Problems

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## 1. INTRODUCTION

In this paper we consider an implicit boundary value problem (BVP) of the form

$$\begin{aligned} F(t, u(t), u'(t) - g(t, u(t))) &= 0 \quad \text{a.e. in} \\ J = [t_o, t_1], \quad B(u(t_o), u(t_1)) &= 0, \end{aligned} \quad (1.1)$$

where the functions  $g: J \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $F: J \times \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $B: \mathbb{R}^2 \rightarrow \mathbb{R}$  do not need to be continuous or monotone in any of their arguments. The study of such types of problems is motivated by recent papers on implicit ordinary differential equations;

$$F(t, u(t), u'(t), \dots, u^{(k)}(t)) = 0 \quad (1.2)$$

under various boundary conditions (cf., e.g., [1–5, 8, 13–29, 31, 32]). There is little known about the existence of solutions of (1.2) even in the case when  $F$  is continuous unless it is possible to solve the equation (1.2) for the highest derivative and then to apply standard existence theory such as Leray–Schauder theory. One approach, taken for instance in [8, 16], is to reduce equation (1.2) to a differential inclusion of the form

$$u^{(k)}(t) \in \Phi(t, u(t), u'(t), \dots, u^{(k-1)}(t)), \quad (1.3)$$

where the multifunction  $\Phi(t, u_0, u_1, \dots, u_{k-1})$  is defined by

$$\Phi(t, u_0, u_1, \dots, u_{k-1}) = \{u_k \in \mathbb{R} \mid F(t, u_0, u_1, \dots, u_{k-1}, u_k) = 0\}.$$

However, the treatment of the inclusion (1.3) requires a “well-behaved” multifunction  $\Phi$  in order to be able to apply recent developments in the theory of multivalued maps and differential inclusions, (cf., e.g., [8]). Moreover, it turns out that it is hard to get explicit conditions on the function  $F$  of the original problem (1.2) which imply the conditions needed for the multifunction  $\Phi$ .

The aim of this paper is to suggest an alternative approach to implicit differential equations involving discontinuous nonlinearities, including also implicitly given boundary conditions, which cannot be handled by the method described above as will be shown by examples given in Section 6 of this paper. In addition, our approach has the advantage that the conditions needed are imposed explicitly on the functions  $F$ ,  $g$ , and  $B$  of the original problem (1.1), and under certain additional regularity assumptions we are able to obtain solutions in a constructive way.

The paper is organized as follows: Section 2 deals with the BVP

$$Lu(t) = Nu(t) \quad \text{a.e. in } J = [t_o, t_1], \quad u(t_o) = Q(u(t_o), u(t_1)), \quad (1.4)$$

where  $L: W \subseteq AC(J) \rightarrow L^1(J)$ ,  $N: W \rightarrow L^1(J)$  and  $Q: \mathbb{R}^2 \rightarrow \mathbb{R}$ . Defining a partial ordering  $\leq$  on the space  $AC(J)$  of all absolutely continuous functions  $u: J \rightarrow \mathbb{R}$  pointwise, and on the space  $L^1(J)$  of all Lebesgue integrable functions  $u: J \rightarrow \mathbb{R}$  a.e. pointwise, we assume that the operator  $L$  is surjective and satisfies a kind of inverse monotonicity condition, and that the operators  $N$  and  $Q$  possess certain monotonicity properties. We prove that (1.4) has extremal solutions within an order interval of  $W$  by using a method of upper and lower solutions and a fixed point result in partially ordered metric spaces based on a generalized iteration principle introduced in [11]. A special feature in our treatment is the use of a partial ordering and a metric of  $W$  which depend on the operator  $L$ . A new procedure is introduced in Section 3 to find such a subset  $W$  of  $AC(J)$  that equation

$$Lu(t) = u'(t) - g(t, u(t)) \quad (1.5)$$

defines an operator  $L: W \rightarrow L^1(J)$  which has properties described above. Classical comparison theorems and maximum principles (cf., e.g., [22]) do not apply since  $g$  is allowed to be discontinuous with respect to both of its variables. In Section 4 the BVP (1.1) is transformed into the general framework (1.4), and the existence of extremal solutions of this transformed equation within an order interval of  $W$  is proved. Finally, in Section 5 we prove our main result concerning the existence of extremal solutions of the BVP (1.1). Dependence of these solutions on the functions

$F$  and  $B$  is also studied. Under certain right-continuity assumptions a method of successive approximations is established to obtain a maximal solution of (1.1). This method is demonstrated in Section 6 by a concrete example whose solutions also have chaotic behavior. After considering special cases we introduce generalizations to the BVP (1.1).

Due to the low regularity assumptions imposed on the problem (1.1) we intend our paper to be an essential generalization to the newly developed theory of discontinuous implicit differential equations. It is our hope that the mathematical tools developed in this paper may be applied to solving modeling problems, e.g., to describe the behavior of a dynamic system whose states are governed by an implicit boundary condition, and when a rate of change of states depends implicitly on the states according to a law which also allows discontinuities.

## 2. ON EXTREMAL SOLUTIONS OF THE BVP (1.4)

In this chapter we study the BVP (1.4), where  $L: W \subseteq AC(J) \rightarrow L^1(J)$ ,  $N: W \rightarrow L^1(J)$  and  $Q: \mathbb{R}^2 \rightarrow \mathbb{R}$ . We assume that  $L$  has the following properties:

(L0) If  $h \in L^1(J)$  and  $x_o \in \mathbb{R}$ , there is  $u \in W$  such that  $Lu = h$  and  $u(t_o) = x_o$ .

(L1) If  $v, w \in W$ ,  $Lv \leq Lw$  and  $v(t_o) \leq w(t_o)$ , then  $v \leq w$ .

An order relation  $\leq$  is defined on the set  $W$  by

$$v \leq w \quad \text{iff} \quad v(t_o) \leq w(t_o) \quad \text{and} \quad Lv \leq Lw. \quad (2.1)$$

Obviously,  $\leq$  is reflexive and transitive, and by condition (L1) also antisymmetric, so that  $\leq$  is a partial ordering.

We say that  $u \in W$  is a *lower solution* of (1.4) if

$$Lu \leq Nu, \quad u(t_o) \leq Q(u(t_o), u(t_1)),$$

and an *upper solution* of (1.4) if the reversed inequalities hold. If equalities hold, we say that  $u$  is a *solution* of (1.4).

We are going to prove that under the hypotheses (L0), (L1), and

(A) (1.4) has a lower solution  $\underline{u}$  and an upper solution  $\bar{u}$ , and  $\underline{u} \leq \bar{u}$ ,

(N) if  $\underline{u} \leq v \leq w \leq \bar{u}$ , then  $Nv \leq Nw$ ,

(Q) if  $\underline{u} \leq v \leq w \leq \bar{u}$ , then  $Q(v(t_o), v(t_1)) \leq Q(w(t_o), w(t_1))$ ,

the BVP (1.4) has extremal solutions in the order interval  $[\underline{u}, \bar{u}] = \{u \in W \mid \underline{u} \leq u \leq \bar{u}\}$ , and that they are monotone nondecreasing with

respect to  $N$  and to  $Q$ . Before the proof we introduce some auxiliary results.

**LEMMA 2.1.** *For each  $u \in [\underline{u}, \bar{u}]$  there is a unique function  $w = Gu$  in  $W$  which satisfies the IVP*

$$LG u = Nu, \quad Gu(t_o) = Q(u(t_o), u(t_1)). \quad (2.2)$$

*The so obtained mapping  $G: [\underline{u}, \bar{u}] \rightarrow [\underline{u}, \bar{u}]$  is monotone nondecreasing.*

*Proof.* Let  $u \in [\underline{u}, \bar{u}]$  be given. In view of (2.1) we have

$$\underline{u}(t_o) \leq u(t_o) \leq \bar{u}(t_o) \quad \text{and} \quad L\underline{u} \leq Lu \leq L\bar{u}. \quad (a)$$

Conditions (A) and (N) imply that

$$L\underline{u} \leq N\underline{u} \leq Nu \leq N\bar{u} \leq L\bar{u}. \quad (b)$$

Since  $Nu \in L^1(J)$ , the IVP (2.2) has by condition (L0) a solution  $w = Gu \in W$ . Condition (L1) implies that this solution is uniquely determined. Thus (2.2) defines a mapping  $G: [\underline{u}, \bar{u}] \rightarrow W$ . It follows from (a), (A), (Q), and (2.2) that

$$\underline{u}(t_o) \leq G\underline{u}(t_o) \leq Gu(t_o) \leq G\bar{u}(t_o) \leq \bar{u}(t_o). \quad (c)$$

In view of (2.2) and inequalities (b) we obtain

$$L\underline{u} \leq LG u \leq L\bar{u}. \quad (d)$$

This inequality, (c) and (2.1) imply that  $\underline{u} \leq Gu \leq \bar{u}$ , whence  $Gu \in [\underline{u}, \bar{u}]$ . This holds for each  $u \in [\underline{u}, \bar{u}]$ , so that  $G[\underline{u}, \bar{u}] \subseteq [\underline{u}, \bar{u}]$ .

If  $u, \hat{u} \in [\underline{u}, \bar{u}]$  and  $u \leq \hat{u}$ , it follows from (2.1), (N), and (Q) that

$$Q(u(t_o), u(t_1)) \leq Q(\hat{u}(t_o), \hat{u}(t_1)) \quad \text{and} \quad Nu \leq N\hat{u}.$$

This and (2.2) imply that

$$Gu(t_o) \leq G\hat{u}(t_o) \quad \text{and} \quad LG u \leq LG \hat{u}, \quad (e)$$

whence  $Gu \leq G\hat{u}$ . Thus  $G$  is monotone nondecreasing. ■

As an immediate consequence of (1.4) and (2.2) we get the following result.

**LEMMA 2.2.** *A function  $u \in [\underline{u}, \bar{u}]$  is a solution of the BVP (1.4) if and only if  $u$  is a fixed point of the operator  $G$ , defined in Lemma 2.1.*

We shall equip the set  $W$  with a metric  $d$ , defined by

$$d(v, w) = |v(t_o) - w(t_o)| + \|Lv - Lw\|_1. \quad (2.3)$$

Obviously,  $W = (W, d, \leq)$  is an ordered metric space, i.e., for each  $v \in W$  the order intervals  $\{u \in W \mid u \leq v\}$  and  $\{u \in W \mid v \leq u\}$  are closed w.r.t. the metric  $d$ .

**LEMMA 2.3.** *If  $(u_n)_{n=0}^\infty$  is a monotone sequence in  $[\underline{u}, \bar{u}]$ , then the sequence  $(Gu_n)_{n=0}^\infty$  converges in  $W$  with respect to the metric  $d$ , defined in (2.3).*

*Proof.* Let  $(u_n)_{n=0}^\infty$  be a monotone sequence in  $[\underline{u}, \bar{u}]$ . Because  $G$  is nondecreasing, then  $(Gu_n)_{n=0}^\infty$  is a monotone sequence in the order interval  $[\underline{u}, \bar{u}]$ . This, (2.1) and (L1) imply that  $(Gu_n(t))_{n=0}^\infty$  is for each  $t \in J$  a monotone sequence in the closed interval  $[\underline{u}(t), \bar{u}(t)]$  of  $\mathbb{R}$ , so that the limits

$$v(t) = \lim_{n \rightarrow \infty} Gu_n(t), \quad t \in J, \quad (a)$$

exist. Since  $(L(Gu_n))_{n=0}^\infty$  is a monotone sequence in the order interval  $[\underline{Lu}, L\bar{u}]$  of  $L^1(J)$ , it converges in  $L^1(J)$  by the monotone convergence theorem. Denoting by  $h$  the limit function, condition (L0) implies an existence of  $u \in W$  such that

$$Lu = h, \quad u(t_o) = v(t_o). \quad (b)$$

Thus

$$\lim_{n \rightarrow \infty} \|L(Gu_n) - Lu\|_1 = 0.$$

This, (a) and (b) imply that  $Gu_n \rightarrow u$  in the metric defined by (2.3). ■

The following result is a special case of Theorem 1.2.2 of [11].

**LEMMA 2.4.** *Let  $[\underline{u}, \bar{u}]$  be an order interval in the ordered metric space  $(W, \leq, d)$ , and let  $G: [\underline{u}, \bar{u}] \rightarrow [\underline{u}, \bar{u}]$  be monotone nondecreasing. If  $(Gu_n)_{n=0}^\infty$  converges in  $W$  whenever  $(u_n)_{n=0}^\infty$  is a monotone sequence in  $[\underline{u}, \bar{u}]$ , then  $G$  has least and greatest fixed points.*

Now we are ready to prove our main existence result for the BVP (1.4).

**THEOREM 2.1.** *Assume that conditions (A), (N), (Q), (L0), and (L1) are valid. Then the BVP (1.4) has a minimal solution  $u_*$  and a maximal solution  $u^*$  in  $[\underline{u}, \bar{u}]$ , in the sense that if  $u \in [\underline{u}, \bar{u}]$  is a solution of (1.4), then  $u \in [u_*, u^*]$ .*

*Proof.* The results of Lemmata 2.1 and 2.3 ensure that the hypotheses of Lemma 2.4 are valid for the operator  $G$  defined by (2.2). Thus  $G$  has a least fixed point  $u_*$  and a greatest fixed point  $u^*$ . In view of Lemma 2.2 this means that  $u_*$  and  $u^*$  are minimal and maximal solutions of the BVP (1.4) in the order interval  $[\underline{u}, \bar{u}]$ . ■

*Remark 2.1.* According to Theorem 1.2.1 of [11] and its dual the extremal solutions  $u_*$  and  $u^*$  of (1.4) in  $[\underline{u}, \bar{u}]$  satisfy

$$\begin{cases} u_* = \max C = \min\{w \in [\underline{u}, \bar{u}] \mid Gw \leq w\}, \\ u^* = \min \hat{C} = \max\{w \in [\underline{u}, \bar{u}] \mid w \leq Gw\}, \end{cases} \quad (2.4)$$

where  $C$  is a well-ordered chain of  $G$ -iterations of  $\underline{u}$  and  $\hat{C}$  is an inversely well-ordered chain of  $G$ -iterations of  $\bar{u}$ . Proposition 1.1.6 of [11] and its dual imply that the chains  $C$  and  $\hat{C}$  are countable. Applying (2.4) we shall prove the following result for the dependence of extremal solutions of (1.4) on the functions  $N$  and  $Q$ .

**PROPOSITION 2.1.** *If conditions (A), (N), (Q), (L0), and (L1) are satisfied, then the extremal solutions of the BVP (1.4) in  $[\underline{u}, \bar{u}]$  are monotone nondecreasing with respect to  $N$  and  $Q$ .*

*Proof.* Assume that conditions (N) and (Q) hold for the functions  $N, \hat{N}: W \rightarrow L^1(J)$  and  $Q, \hat{Q}: \mathbb{R}^2 \rightarrow \mathbb{R}$ , respectively. Assume also that

$$Nu \leq \hat{N}u \quad \text{and} \quad Q(u(t_o), u(t_1)) \leq \hat{Q}(u(t_o), u(t_1)) \quad \text{for all } u \in W. \quad (a)$$

Moreover, we shall assume an existence of  $\underline{u}, \bar{u} \in W$ ,  $\underline{u} \leq \bar{u}$ , so that they are lower and upper solutions of both the BVP (1.4) and

$$Lu = \hat{N}u, \quad u(t_o) = \hat{Q}(u(t_o), u(t_1)). \quad (2.5)$$

Thus problems (1.4) and (2.5) have by Theorem 2.1 minimal solutions  $u_*, \hat{u}_*$  and maximal solutions  $u^*, \hat{u}^*$  in the order interval  $[\underline{u}, \bar{u}]$ .

It follows from (2.2), (a) and (2.5) that

$$\begin{cases} LG\hat{u}_* = N\hat{u}_* \leq \hat{N}\hat{u}_* = L\hat{u}_*, \\ G\hat{u}_*(t_o) = Q(\hat{u}_*(t_o), \hat{u}_*(t_1)) \leq \hat{Q}(\hat{u}_*(t_o), \hat{u}_*(t_1)) = \hat{u}_*(t_o). \end{cases}$$

These relations and (2.1) imply that  $G\hat{u}_* \leq \hat{u}_*$ . Since  $\hat{u}_* \in [\underline{u}, \bar{u}]$ , it then follows from the first formula of (2.4) that  $u_* \leq \hat{u}_*$ .

Similarly, it can be shown, by applying the second formula of (2.4), that  $u^* \leq \hat{u}^*$ , which concludes the proof. ■

*Remark 2.2.* In the study of the operator equation of the form  $Lu = Nu$  it is usually assumed that  $L$  is a linear operator, and that  $N$  is continuous

(cf. [22]). These assumptions would yield essentially more restrictive hypotheses in the next applications.

### 3. DEFINITION OF THE OPERATOR $L$

To construct a subset  $W$  of  $AC(J)$  so that (1.5) defines an operator  $L: W \rightarrow L^1(J)$  which has properties (L0) and (L1), assume that a function  $g: J \times \mathbb{R} \rightarrow \mathbb{R}$  satisfies the following hypotheses:

(g0) For each  $x \in \mathbb{R}$  the function  $g(\cdot, x)$  is measurable, and  $\limsup_{y \rightarrow x-} g(t, y) \leq g(t, x) = \lim_{y \rightarrow x+} g(t, y)$  for almost all (a.a.)  $t \in J$ .

(g1)  $|g(t, x)| \leq p_1(t)\psi(|x|)$  for all  $x \in \mathbb{R}$  and a.a.  $t \in J$ , where  $p_1 \in L^1_+(J)$ , the function  $\psi: \mathbb{R}_+ \rightarrow (0, \infty)$  is monotone nondecreasing and  $\int_0^\infty dx/\psi(x) = \infty$ .

The following existence and comparison result for explicit IVPs is a consequence of [12, Lemma 3.1].

**LEMMA 3.1.** *If conditions (g0) and (g1) are valid, and if  $h \in L^1(J)$  and  $x_o \in \mathbb{R}$ , then the IVP*

$$u'(t) = g(t, u(t)) + h(t) \quad \text{for a.a. } t \in J, \quad u(t_o) = x_o \quad (3.1)$$

*has a maximal solution  $u$ . Moreover, if  $v \in AC(J)$  is a lower solution of (3.1), i.e.,*

$$v'(t) \leq g(t, v(t)) + h(t) \quad \text{for a.a. } t \in J, \quad v(t_o) \leq x_o, \quad (3.2)$$

*then  $v(t) \leq u(t)$  on  $J$ .*

*Proof.* For each  $h \in L^1(J)$  the functions  $q: \mathbb{R} \rightarrow (0, \infty)$  and  $f: J \times \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by

$$q(x) := 1, \quad f(t, x, y) := g(t, x) + h(t), \quad t \in J, \quad x, y \in \mathbb{R}$$

satisfy the hypotheses of [12, Lemma 3.1], which implies the conclusions. ■

When  $u \in AC(J)$ , define  $u'(t) = 0$  at those points  $t \in J$  where  $u$  is not differentiable.

**LEMMA 3.2.** *Assume that  $g$  satisfies conditions (g0) and (g1). Defining*

$$\begin{cases} W := \{\text{maximal solutions of (3.1) for all } h \in L^1(J) \text{ and } x_o \in \mathbb{R}\}, \\ Lu(t) := u'(t) - g(t, u(t)), \quad u \in W, \quad t \in J, \end{cases} \quad (3.3)$$

*we obtain a mapping  $L: W \rightarrow L^1(J)$  which has properties (L0) and (L1).*

*Proof.* Property (L0) is a consequence of Lemma 3.1. To prove (L1), assume that  $v, w \in W$  satisfy  $Lv \leq Lw$  and  $v(t_o) \leq w(t_o)$ . Denoting  $h_1 = Lv$ ,  $h_2 = Lw$ ,  $y_o = v(t_o)$  and  $x_o = w(t_o)$  it follows that  $w$  is a maximal solution of the IVP

$$u'(t) = g(t, u(t)) + h_2(t) \quad \text{for a.a. } t \in J, \quad u(t_o) = x_o. \quad (a)$$

Because  $h_1 \leq h_2$  and  $y_o \leq x_o$ , and since the function  $v$  is a solution of the IVP

$$u'(t) = g(t, u(t)) + h_1(t) \quad \text{for a.a. } t \in J, \quad u(t_o) = y_o,$$

then  $v$  is a lower solution of (a). This implies by Lemma 3.1 that  $v \leq w$ , so that condition (L1) is valid. ■

*Remarks 3.1.* Conditions (g0) and (g1) do not ensure that the IVP (3.1) has a unique solution so that, in general, the operator  $L$  does not possess property (L1). For this reason we restrict the domain of  $L$  to a subset  $W$  of  $AC(J)$  introduced by (3.3) to obtain an operator with the property (L1). Instead of maximal solutions of (3.1), its minimal solutions can be chosen in the definition (3.3) of  $W$ , because dual results of Lemma 3.1 hold by [12, Lemma 3.1].

#### 4. AN EXISTENCE RESULT

Throughout this chapter we assume that conditions (g0) and (g1) are satisfied, and that  $W$  and  $L$  are defined by (3.3). We are going to prove that the BVP

$$\begin{cases} Lu(t) = Nu(t) := Lu(t) - (\mu \cdot F)(t, u(t), Lu(t)), & \text{a.e. in } J, \\ u(t_o) = Q(u(t_o), u(t_1)) := u(t_o) - (v \cdot B)(u(t_o), u(t_1)), \end{cases} \quad (4.1)$$

has a maximal solution in an order interval of  $W$  if the functions  $F: J \times \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $B: \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfy the following hypotheses:

(F0) There is  $\mu: J \times \mathbb{R}^2 \rightarrow (0, \infty)$  such that  $\mu \cdot F$  is sup-measurable and  $(t, x, y) \mapsto y - (\mu \cdot F)(t, x, y)$  is monotone nondecreasing w.r.t.  $x$  and  $y$  for a.a.  $t \in J$ .

(F1)  $|y - (\eta \cdot F)(t, x, y)| \leq p_2(t)\psi(|x|) + \lambda(t)|y|$ , for a.a.  $t \in J$  and all  $x, y \in \mathbb{R}$ , where  $\eta: J \times \mathbb{R}^2 \rightarrow (0, \infty)$ ,  $\lambda: J \rightarrow [0, 1]$ ,  $p_2/(1 - \lambda) \in L^1_+(J)$ ,  $\psi: \mathbb{R}_+ \rightarrow (0, \infty)$  is monotone nondecreasing and  $\int_o^\infty dx/\psi(x) = \infty$ .

(B0) There is  $v: \mathbb{R}^2 \rightarrow (0, \infty)$  such that  $(x, y) \mapsto x - (v \cdot B)(x, y)$  is monotone nondecreasing w.r.t. both variables.



(B1)  $|x - (\gamma \cdot B)(x, y)| \leq c|x| + d$  for all  $x, y \in \mathbb{R}$ , where  $\gamma: \mathbb{R}^2 \rightarrow (0, \infty)$ ,  $c \in [0, 1)$  and  $d \in \mathbb{R}_+$ .

It should be noted that in view of the above hypotheses the functions  $F$  and  $B$  need not to be continuous or monotone in any of their arguments. In the next two lemmata we construct functions  $\underline{u}, \bar{u} \in AC(J)$  and show that they are lower and upper solutions of the BVP (4.1).

LEMMA 4.1. *Let  $z \in AC(J)$  be the solution of the IVP*

$$z'(t) = \left( p_1(t) + \frac{p_2(t)}{1 - \lambda(t)} \right) \psi(z(t)), \quad z(t_o) = \frac{d}{1 - c}, \quad (4.2)$$

and let the functions  $\underline{u}$  and  $\bar{u}$  of  $W$  be defined by

$$\begin{cases} L\underline{u}(t) = -\frac{p_2(t) \psi(z(t))}{1 - \lambda(t)}, & \text{for a.a. } t \in J, \quad \underline{u}(t_o) = -\frac{d}{1 - c}, \\ L\bar{u}(t) = \frac{p_2(t) \psi(z(t))}{1 - \lambda(t)}, & \text{for a.a. } t \in J, \quad \bar{u}(t_o) = \frac{d}{1 - c}. \end{cases} \quad (4.3)$$

Then  $|\underline{u}(t)| \leq z(t)$  and  $|\bar{u}(t)| \leq z(t)$  for all  $t \in J$ .

*Proof.* Conditions (F1) and (g1) imply by [11, Lemma 1.5.3] that the IVP (4.2) has a unique solution  $z \in AC(J)$ . It follows from (3.3) and (4.3) that if  $u = \underline{u}$  or  $u = \bar{u}$ , then

$$|u'(t) - g(t, u(t))| = \frac{p_2(t)}{1 - \lambda(t)} \psi(z(t)) \quad \text{for a.a. } t \in J. \quad (a)$$

This and condition (g1) imply that

$$\begin{aligned} |u'(t)| &\leq |u'(t) - g(t, u(t))| + |g(t, u(t))| \\ &\leq |u'(t) - g(t, u(t))| + p_1(t) \psi(|u(t)|) \\ &= \frac{p_2(t) \psi(z(t))}{1 - \lambda(t)} + p_1(t) \psi(|u(t)|). \end{aligned}$$

Denoting  $v(t) = |u(t)|$ ,  $t \in J$ , we then have

$$v'(t) \leq \frac{p_2(t) \psi(z(t))}{1 - \lambda(t)} + p_1(t) \psi(v(t)) \quad \text{for a.a. } t \in J, \quad v(t_o) = \frac{d}{1 - c}. \quad (b)$$

Because  $z$  is the solution of the IVP (4.2), then it is a solution of the IVP

$$z'(t) = \frac{p_2(t) \psi(z(t))}{1 - \lambda(t)} + p_1(t) \psi(z(t)) \quad \text{for a.a. } t \in J, \quad y(t_o) = \frac{d}{1 - c}. \quad (c)$$

Denoting  $y = \max\{v, z\}$ , it follows from (b) and (c) by the monotonicity of  $\psi$  that

$$y'(t) \leq \left( \frac{p_2(t)}{1 - \lambda(t)} + p_1(t) \right) \psi(y(t)) \quad \text{for a.a. } t \in J, \quad y(t_o) = \frac{d}{1 - c}.$$

In view of this, (4.2) and [11, Lemma 1.5.3] we see that  $y(t) \leq z(t)$  on  $J$ . Thus  $v(t) \leq z(t)$ , i.e.,  $|u(t)| \leq z(t)$  on  $J$ , which proves the assertion. ■

**LEMMA 4.2.** *Let  $\underline{u}$  and  $\bar{u}$  be the functions given by (4.3). Then  $\underline{u}$  is a lower solution,  $\bar{u}$  is an upper solution of the BVP (4.1), and  $\underline{u} \leq \bar{u}$ .*

*Proof.* Applying condition (F1), the result of Lemma 4.1 and (4.3) we get

$$\begin{aligned} (\eta \cdot F)(t, \bar{u}(t), L\bar{u}(t)) &\geq L\bar{u}(t) - |L\bar{u}(t) - (\eta \cdot F)(t, \bar{u}(t), L\bar{u}(t))| \\ &\geq L\bar{u}(t) - p_2(t) \psi(|\bar{u}(t)|) - \lambda(t) L\bar{u}(t) \\ &\geq L\bar{u}(t) - p_2(t) \psi(z(t)) - \lambda(t) L\bar{u}(t) \\ &= (1 - \lambda(t)) L\bar{u}(t) - p_2(t) \psi(z(t)) = 0, \end{aligned}$$

and

$$\begin{aligned} (\eta \cdot F)(t, \underline{u}(t), L\underline{u}(t)) &\leq L\underline{u}(t) + |L\underline{u}(t) - (\eta \cdot F)(t, \underline{u}(t), L\underline{u}(t))| \\ &\leq L\underline{u}(t) + p_2(t) \psi(|\underline{u}(t)|) - \lambda(t) L\underline{u}(t) \\ &\leq L\underline{u}(t) + p_2(t) \psi(z(t)) - \lambda(t) L\underline{u}(t) \\ &= (1 - \lambda(t)) L\underline{u}(t) + p_2(t) \psi(z(t)) = 0 \end{aligned}$$

a.e. in  $J$ . Since  $\bar{u}(t_o) \geq 0$  and  $\underline{u}(t_o) \leq 0$ , it follows from condition (B1) that

$$\begin{aligned} (\gamma \cdot B)(\bar{u}(t_o), \bar{u}(t_1)) &= \bar{u}(t_o) - (\bar{u}(t_o) - (\gamma \cdot B)(\bar{u}(t_o), \bar{u}(t_1))) \\ &\geq \bar{u}(t_o) - c\bar{u}(t_o) - d = 0, \\ (\gamma \cdot B)(\underline{u}(t_o), \underline{u}(t_1)) &= \underline{u}(t_o) - (\underline{u}(t_o) - (\gamma \cdot B)(\underline{u}(t_o), \underline{u}(t_1))) \\ &\leq \underline{u}(t_o) - c\underline{u}(t_o) + d = 0. \end{aligned}$$

Because  $\eta$  and  $\gamma$  are positive-valued, the above proofs imply that

$$\begin{cases} F(t, \underline{u}(t), L\underline{u}(t)) \leq 0, & \text{and } F(t, \bar{u}(t), L\bar{u}(t)) \geq 0, & \text{a.e. in } J, \\ B(\underline{u}(t_o), \underline{u}(t_1)) \leq 0, & \text{and } B(\bar{u}(t_o), \bar{u}(t_1)) \geq 0. \end{cases}$$

Thus  $\underline{u}$  is a lower solution of (4.1) and  $\bar{u}$  is its upper solution. The relation  $\underline{u} \leq \bar{u}$  is a direct consequence of (4.3) and (2.1).

As a consequence of Lemmata 3.2 and 4.2 and Theorem 2.1 we get the following result.

**PROPOSITION 4.1.** *Let the hypotheses (F0), (F1), (g0), (g1), (B0), and (B1) be satisfied, and let  $\underline{u}, \bar{u}$  be defined by (4.3). Then the BVP (4.1) has a maximal solution  $u^*$  in the order interval  $[\underline{u}, \bar{u}]$ , and  $u^*$  is also a solution of the BVP (1.1).*

*Proof.* It follows from Lemma 4.2 that condition (A) is valid. Conditions (N) and (Q) are immediate consequences of the definitions of  $N$  and  $Q$  given in (4.1) and hypotheses (F0) and (B0). Because conditions (L0) and (L1) are valid by Lemma 3.2, it follows from Theorem 2.1 that the BVP (4.1) has a maximal solution  $u^*$  in the order interval  $[\underline{u}, \bar{u}]$ .  $u^*$  is also a solution of the BVP (1.1), since the definitions (3.3) and (4.1) of  $L$ ,  $N$ , and  $Q$  and the fact that  $\mu$  and  $\nu$  are positive-valued imply that each solution of (4.1) is also a solution of the BVP (1.1). ■

## 5. MAIN RESULTS

Now we are ready to prove our main existence result for the BVP (1.1).

**THEOREM 5.1.** *Assume that the hypotheses (F0), (F1), (g0), (g1), (B0), and (B1) are satisfied. Then the BVP (1.1) has extremal solutions  $u_*$  and  $u^*$  in the sense that if  $u \in AC(J)$  is any solution of (1.1), then  $u_*(t) \leq u(t) \leq u^*(t)$  for all  $t \in J$ .*

*Proof.* We shall prove the existence of a maximal solution. Let  $W$  and  $L$  be defined by (3.3). In view of Proposition 4.1 the BVP (4.1) has a maximal solution  $u^*$  in the order interval  $[\underline{u}, \bar{u}]$ , and  $u^*$  is also a solution of the BVP (1.1). To prove that  $u^*$  is a maximal solution of (1.1), let  $u \in AC(J)$  be any solution of (1.1). Applying condition (F1) we get for a.a.  $t \in J$ ,

$$\begin{aligned} |u'(t) - g(t, u(t))| &= |u'(t) - g(t, u(t)) - (\eta \cdot F)(t, u(t), u'(t) - g(t, u(t)))| \\ &\leq p_2(t) \psi(|u(t)|) + \lambda(t) |u'(t) - g(t, u(t))|, \end{aligned}$$

so that

$$|u'(t) - g(t, u(t))| \leq \frac{p_2(t)}{1 - \lambda(t)} \psi(|u(t)|) \quad \text{for a.a. } t \in J. \quad (a)$$

This and condition (g1) imply

$$\begin{aligned} |u'(t)| &\leq |u'(t) - g(t, u(t))| + |g(t, u(t))| \\ &\leq |u'(t) - g(t, u(t))| + p_1(t) \psi(|u(t)|) \\ &\leq \frac{p_2(t) \psi(|u(t)|)}{1 - \lambda(t)} + p_1(t) \psi(|u(t)|) = \left( p_1(t) + \frac{p_2(t)}{1 - \lambda(t)} \right) \psi(|u(t)|) \end{aligned}$$

for a.a.  $t \in J$ . In view of condition (B1) we obtain

$$|u(t_o)| = |u(t_o) - (\gamma \cdot B)(u(t_o), u(t_1))| \leq c |u(t_o)| + d,$$

so that  $|u(t_o)| \leq d/(1 - c)$ . Thus

$$\begin{aligned} |u(t)| &\leq |u(t_o)| + \int_{t_o}^t |u'(s)| \, ds \\ &\leq \frac{d}{1 - c} + \int_{t_o}^t \left( p_1(s) + \frac{p_2(s)}{1 - \lambda(s)} \right) \psi(|u(s)|) \, ds, \quad t \in J. \end{aligned}$$

Noticing that  $z$  is a solution of the IVP (4.2), this implies by [11, Lemma 1.5.3] that  $|u(t)| \leq z(t)$  on  $J$ .

Denote

$$h_u(t) := u'(t) - g(t, u(t)) - (\mu \cdot F)(t, u(t), u'(t) - g(t, u(t))), \quad t \in J. \quad (b)$$

Because  $u$  is a solution of (1.1), it follows from (a) and (b) by using property  $|u(t)| \leq z(t)$  on  $J$  and the monotonicity of  $\psi$ , that

$$|h_u(t)| = |u'(t) - g(t, u(t))| \leq \frac{p_2(t) \psi(z(t))}{1 - \lambda(t)} \quad \text{for a.a. } t \in J. \quad (c)$$

Hence,  $h_u \in L^1(J)$ . Let  $\hat{u}$  be the maximal solution of (3.1) with  $h = h_u$  and  $x_o = Q(u(t_o), u(t_1))$ . Since  $u$  is also a solution of this problem and (1.1), then

$$u(t) \leq \hat{u}(t) \quad \text{on } J \quad \text{and} \quad L\hat{u}(t) = h_u(t) = u'(t) - g(t, u(t)), \quad \text{a.e. in } J. \quad (d)$$

In view of (b), (d) and conditions (F0) and (B0) we then have

$$L\hat{u}(t) = h_u(t) \leq L\hat{u}(t) - (\mu \cdot F)(t, \hat{u}(t), L\hat{u}(t)) = N\hat{u}(t) = LG\hat{u}(t) \quad \text{a.e. in } J,$$

and

$$\begin{aligned}\hat{u}(t_o) &= Q(u(t_o), u(t_1)) = u(t_o) - (v \cdot B)(u(t_o), u(t_1)) \\ &\leq \hat{u}(t_o) - (v \cdot B)(\hat{u}(t_o), \hat{u}(t_1)) = Q(\hat{u}(t_o), \hat{u}(t_1)) = G\hat{u}(t_o).\end{aligned}$$

These inequalities imply that  $\hat{u} \leq G\hat{u}$ . It follows from (c) and (d) that

$$|L\hat{u}(t)| = |h_u(t)| \leq \frac{p_2(t) \psi(z(t))}{1 - \lambda(t)}$$

for a.a.  $t \in J$ . This and  $|\hat{u}(t_o)| = |Q(u(t_o), u(t_1))| = |u(t_o)| \leq d/(1 - c)$  imply by (4.3) and (2.1) that  $\underline{u} \leq \hat{u} \leq \bar{u}$ . Thus  $\hat{u} \leq u^*$  by (2.4), so that  $\hat{u}(t) \leq u^*(t)$  on  $J$  by condition (L1). This and (d) imply that  $u(t) \leq u^*(t)$  on  $J$ , whence  $u^*$  is the maximal solution of the BVP (1.1). The proof for the existence of the minimal solution is similar. ■

As a consequence of Theorem 5.1, Proposition 2.1, and the definitions (4.1) of  $N$  and  $Q$  we get the following result.

**PROPOSITION 5.1.** *If the hypotheses of Theorem 5.1 hold, then the extremal solutions of the BVP (1.1) are monotone nonincreasing with respect to  $F$  and to  $B$ .*

Consider next the IVP (1.1) in the case when right-continuity hypotheses hold for the function  $\mu \cdot F$  in its last two arguments, and for  $v \cdot B$  in its both arguments. Given a sequence  $(x_n)$  in  $\mathbb{R}$  converging to  $x$ , denote  $x_n \searrow x$  if  $(x_n)$  is monotone nonincreasing.

**LEMMA 5.1.** *Assume that  $g: J \times \mathbb{R} \rightarrow \mathbb{R}$  has properties (g0) and (g1). Then the operator  $L: W \rightarrow L^1(J)$ , where  $W$  and  $L$  are defined by (3.3), has property*

(L2) *If  $(u_n)$  is a sequence in  $[\underline{u}, \bar{u}]$  such that  $u_{n+1} \leq u_n$  for each  $n$ , then  $(u_n)$  converges uniformly on  $J$  to a function  $u \in W$  and  $Lu_n(t) \searrow Lu(t)$  a.e. in  $J$ .*

*Proof.* Assume  $(u_n)$  is a sequence in  $[\underline{u}, \bar{u}]$ , and that  $u_{n+1} \leq u_n$  for each  $n$ . In view of (2.1) and property (L1),  $(u_n(t))$  is a monotone nonincreasing sequence in  $[\underline{u}(t), \bar{u}(t)]$  for each  $t \in J$ , and  $(Lu_n(t))$  is a monotone nonincreasing sequence in  $[L\underline{u}(t), L\bar{u}(t)]$  for a.a.  $t \in J$ . Thus the limits

$$u(t) := \lim_{n \rightarrow \infty} u_n(t), \quad t \in J \quad \text{and} \quad h(t) := \lim_{n \rightarrow \infty} Lu_n(t) \quad \text{a.e. in } J \quad (\text{a})$$

exist. From

$$u'_n(t) - g(t, u_n(t)) = Lu_n(t), \quad n \in \mathbb{N}, \quad t \in J$$

it follows by integration that

$$u_n(t) = u_n(t_o) + \int_{t_o}^t (g(s, u_n(s)) + Lu_n(s)) ds, \quad n \in \mathbb{N}, \quad t \in J. \quad (b)$$

The above assumptions, (a) and condition (g0) imply that

$$\lim_{n \rightarrow \infty} (g(t, u_n(t)) + Lu_n(t)) = g(t, u(t)) + h(t) \quad \text{for a.a. } t \in J. \quad (c)$$

Since  $(u_n)$  belongs to  $[u, \bar{u}]$ , then  $M = \sup\{|u_n(t)| \mid t \in J, n \in \mathbb{N}\} < \infty$ . It then follows from condition (g1) that

$$|g(t, u_n(t))| \leq p_1(t) \psi(M) \quad \text{for a.a. } t \in J \quad \text{and for all } n \in \mathbb{N}.$$

In view of this, (a), (c) and the dominated convergence it follows from (b) when  $n \rightarrow \infty$  that

$$u(t) = u(t_o) + \int_{t_o}^t (g(s, u(s)) + h(s)) ds, \quad t \in J.$$

This implies that  $u \in AC(J)$ , and that  $u'(t) - g(t, u(t)) = h(t)$  for a.a.  $t \in J$ . Denoting  $u(t_o) = x_o$ , then  $u$  is a solution of the IVP

$$u'(t) = g(t, u(t)) + h(t) \quad \text{for a.a. } t \in J, \quad u(t_o) = x_o. \quad (3.1)$$

Denote by  $\hat{u}$  the maximal solution of (3.1). Since  $L\hat{u}(t) = h(t) \leq Lu_n(t)$  for a.a.  $t \in J$ , and  $\hat{u}(t_o) = x_o \leq u_n(t_o)$ , it follows from property (L1) that  $\hat{u}(t) \leq u_n(t)$ ,  $t \in J$ . This inequality and (a) imply when  $n \rightarrow \infty$  that  $\hat{u}(t) \leq u(t)$  on  $J$ . The reverse inequality holds since  $u$  is a solution of (3.1) and  $\hat{u}$  is its maximal solution. Thus  $u = \hat{u} \in W$ , and  $Lu = h$ . This and (a) imply that  $Lu_n(t) \searrow Lu(t)$  a.e. on  $J$ . Since  $u$  is continuous, then the monotone sequence  $(u_n)$  converges uniformly on  $J$  to  $u$ , so that condition (L2) is valid. ■

The result of Lemma 5.1 will next be applied to prove the following result.

**PROPOSITION 5.2.** *Assume that conditions (F0), (F1), (B0), (B1), (g0), and (g1) hold. Then the successive approximations  $u_n$ ,  $n \in \mathbb{N}$ , defined by  $u_o = \bar{u}$ , with  $\bar{u}$  given by (4.3), and*

$$\begin{cases} Lu_{n+1}(t) = Lu_n(t) - (\mu \cdot F)(t, u_n(t), Lu_n(t)), & \text{a.e. in } J, \\ u_{n+1}(t_o) = u_n(t_o) - (\nu \cdot B)(u_n(t_o), u_n(t_1)), & n \in \mathbb{N}, \end{cases} \quad (5.1)$$

converge uniformly on  $J$  to a maximal solution of the BVP (1.1) if  $(\mu \cdot F)(t, x_n, y_n) \rightarrow (\mu \cdot F)(t, x, y)$  for a.a.  $t \in J$  and  $(v \cdot B)(x_n, y_n) \rightarrow (v \cdot B)(x, y)$  whenever  $x_n \searrow x$  and  $y_n \searrow y$ .

*Proof.* The sequence  $(u_n)_{n=0}^\infty$  defined by (5.1) equals the iteration sequence  $(G^n \bar{u})_{n=0}^\infty$ , where  $G$  is defined by (2.2), (4.1). Since  $G: [\underline{u}, \bar{u}] \rightarrow [\underline{u}, \bar{u}]$  is nondecreasing by Lemma 2.1, then (5.1) defines a monotone nonincreasing sequence  $(u_n)_{n=0}^\infty$  in  $[\underline{u}, \bar{u}]$ . This and property (L2) imply an existence of  $u^+ \in AC(J)$  such that

$$u_n(t) \searrow u^+(t) \quad \text{uniformly on } J \quad \text{and} \quad Lu_n(t) \searrow L^+(t) \quad \text{a.e. in } J. \quad (\text{a})$$

The given continuity hypotheses, (a), conditions (F0), (B0), (4.1), and (5.1) imply that

$$\begin{cases} Lu_{n+1}(t) = Nu_n(t) \searrow Nu^+(t) & \text{for a.a. } t \in J, \\ u_{n+1}(t_o) = Q(u_n(t_o), u_n(t_1)) \searrow Q(u^+(t_o), u^+(t_1)). \end{cases} \quad (\text{b})$$

In view of (a) and (b) we get

$$Lu^+(t) = Nu^+(t) \quad \text{for a.a. } t \in J, \quad u^+(t_o) = Q(u^+(t_o), u^+(t_1)). \quad (\text{c})$$

This implies that  $u^+$  is a solution of the BVP (4.1). If  $u$  is any solution of (4.1) in the order interval  $[\underline{u}, \bar{u}]$ , it is a fixed point of  $G$ , so that  $u = Gu \leq \bar{u}$ . Since  $G$  is monotone nondecreasing in  $[\underline{u}, \bar{u}]$  by Lemma 2.1, it is then easy to see by induction that  $u \leq u_n$  for each  $n \in \mathbb{N}$ , which implies, as  $n \rightarrow \infty$ , that  $u \leq u^+$ . This and (L1) imply that  $u^* \leq u^+$ , where  $u^*$  is the maximal solution of the BVP (4.1) in  $[\underline{u}, \bar{u}]$ . By the proof of Theorem 5.1,  $u^*$  is a maximal solution of the BVP (1.1). Because  $u^+$  is also a solution of (1.1), then  $u^+ \leq u^*$ , so that  $u^+ = u^*$ . ■

*Remarks 5.1.* The proof of Theorem 5.1 shows that a maximal solution of the BVP (4.1) with respect to the new partial ordering defined by (2.1) is in fact a maximal solution of the BVP (1.1) with respect to the natural (pointwise) partial ordering of functions. According to the authors' knowledge Theorem 5.1 and Propositions 5.1 and 5.2 are the first ones to handle the case where  $g(t, \cdot)$  in (1.1) is discontinuous and solutions of (3.1) are not uniquely determined. Because classical comparison and maximum principles do not work in this case, our method to find extremal solutions of (1.1) among extremal solutions of (3.1) seems to be an essential step in the proofs of these results. This difficulty does not occur in papers [6, 9, 10, 30] where a generalized iteration method is applied to implicit differential equations involving discontinuities.

## 6. SPECIAL CASES, GENERALIZATIONS AND EXAMPLES

Consider next the BVP

$$\begin{aligned} u'(t) &= g(t, u(t)) + f(t, u(t), u'(t) - g(t, u(t))) \quad \text{a.e. in } J, \\ u(t_0) &= C(u(t_0), u(t_1)), \end{aligned} \quad (6.1)$$

where the function  $g: J \times \mathbb{R} \rightarrow \mathbb{R}$  has properties (g0) and (g1), and the functions  $f: J \times \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $C: \mathbb{R}^2 \rightarrow \mathbb{R}$  are assumed to have the following properties:

(f0)  $f$  is sup-measurable, and there is a measurable function  $\alpha: J \rightarrow \mathbb{R}_+$  such that  $f(t, x, y) + \alpha(t)y$  is monotone nondecreasing in  $x$  and  $y$  for a.a.  $t \in J$ .

(f1)  $|f(t, x, y)| \leq p_2(t) \psi(|x|) + \lambda(t) |y|$  for a.a.  $t \in J$  and for all  $x, y \in \mathbb{R}$ , where  $p_2: J \rightarrow \mathbb{R}_+$  and  $\lambda: J \rightarrow [0, 1)$ ,  $p_2/(1 - \lambda) \in L^1(J)$ ,  $\psi: \mathbb{R}_+ \rightarrow (0, \infty)$  is monotone nondecreasing and  $\int_0^\infty dx/\psi(x) = \infty$ .

(C0)  $C(x, y) + \beta x$  is monotone nondecreasing in  $x$  and  $y$  for some  $\beta \geq 0$ .

(C1) There exist  $c \in [0, 1)$  and  $d \geq 0$  such that  $|C(x, y)| \leq c|x| + d$  for  $x, y \in \mathbb{R}$ .

As an application of Theorem 5.1 and Proposition 5.1 we get the following results.

**PROPOSITION 6.1.** *Assume that conditions (f0), (f1), (g0), (g1), (C0), and (C1) are valid. Then the BVP (6.1) has extremal solutions, and they are monotone nondecreasing with respect to  $f$  and  $C$ .*

*Proof.* Conditions (f0), (f1), (C0) and (C1) imply that the functions  $F: J \times \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $B: \mathbb{R}^2 \rightarrow \mathbb{R}$ , defined by

$$F(t, x, y) = y - f(t, x, y), \quad B(x, y) = x - C(x, y), \quad t \in J, \quad x, y \in \mathbb{R} \quad (6.2)$$

have properties (F0), (F1), (B0) and (B1) when

$$\mu(t, x, y) = \frac{1}{\alpha(t) + 1}, \quad \nu(x, y) \equiv \frac{1}{\beta + 1}, \quad \eta(t, x, y) \equiv 1, \quad \gamma(x, y) \equiv 1. \quad (6.3)$$

Thus the BVP (1.1), with  $F$  and  $B$  defined by (6.2), has by Theorem 5.1 extremal solutions  $u_*$  and  $u^*$ , and they are by Proposition 5.1 monotone



nonincreasing w.r.t.  $F$  and  $B$ . In view of (6.2),  $u_*$  and  $u^*$  are then extremal solutions of (6.1), and they are monotone nondecreasing w.r.t.  $f$  and  $C$ . ■

When the last variable of  $f$  is dropped we get, as a consequence of Proposition 6.1, the following result concerning explicit BVP's.

**COROLLARY 6.1.** *Assume that conditions (g0), (g1), (C0) and (C1) are valid, and that*

(f2)  *$f: J \times \mathbb{R} \rightarrow \mathbb{R}$  is sup-measurable,  $f(t, \cdot)$  is monotone nondecreasing for a.a.  $t \in J$  and  $|f(t, x)| \leq p_2(t) \psi(|x|)$  for a.a.  $t \in J$  and for all  $x \in \mathbb{R}$ , where  $p_2 \in L^1(J)$ ,  $\psi: \mathbb{R}_+ \rightarrow (0, \infty)$  is monotone nondecreasing, and  $\int_0^\infty dx/\psi(x) = \infty$ .*

*Then the BVP*

$$u'(t) = g(t, u(t)) + f(t, u(t)) \quad \text{a.e. in } J, \quad u(t_o) = C(u(t_o), u(t_1))$$

*has extremal solutions, and they are monotone nondecreasing w.r.t.  $f$  and  $C$ .*

The next result is a special case of Proposition 5.2.

**PROPOSITION 6.2.** *Assume that conditions (f0), (f1), (g0), (g1), (C0), are (C1) are valid. If  $u_o = \bar{u}$ , where  $\bar{u}$  is given by (4.3), then the sequence  $(u_n)_{n=o}^\infty$ , defined by*

$$\begin{cases} Lu_{n+1}(t) = \frac{f(t, u_n(t), Lu_n(t)) + \alpha(t) Lu_n(t)}{\alpha(t) + 1} & \text{a.e. in } J, \\ u_{n+1}(t_o) = \frac{C(u_n(t_o), u_n(t_1)) + \beta u_n(t_o)}{1 + \beta}, & n \in \mathbb{N}, \end{cases} \quad (6.4)$$

*converges uniformly on  $J$  to a maximal solution of (6.1) if  $f(t, x_n, y_n) \rightarrow f(t, x, y)$  for a.a.  $t \in J$  and  $C(x_n, y_n) \rightarrow C(x, y)$  whenever  $x_n \searrow x$  and  $y_n \searrow y$ .*

**EXAMPLE 6.1.** Consider the BVP

$$\begin{cases} u'(t) = H(u(t) - t) + \left[ 2 - \frac{3}{2}t + \frac{u(t)}{8} \right] \\ \quad + \frac{[u'(t) - H(u(t) - t)]}{2} & \text{a.e. in } J = [0, 1], \\ u(0) = \frac{[u(1)]}{1 + |[u(1)]|} - 1, \end{cases} \quad (6.5)$$

where  $H$  is the Heaviside function:

$$H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

and  $[x]$  denotes the greatest integer less than or equal to  $x$ . It is easy to see that the hypotheses of Proposition 6.1 hold. Thus the BVP (6.5) has extremal solutions. These and other solutions of (6.5) can be found by inspection as follows: We can restrict our considerations to the set  $\Omega = J \times [-3, 3]$ , where the differential equation of (6.5) can be reduced to an inclusion

$$u'(t) \in \Phi(t, u(t)) = \begin{cases} \{3, \frac{5}{2}\} & \text{if } \max\{t, 12t-8\} \leq u(t) \leq 3, & t \in J, \\ \{2, \frac{3}{2}\} & \text{if } 12t-8 \leq u(t) < t, & t \in J, \\ \{1, \frac{1}{2}\} & \text{if } t \leq u(t) < 12t-8, & t \in J, \\ \{0, -\frac{1}{2}\} & \text{if } -3 \leq u(t) < \min\{t, 2t-8\}, & t \in J, \end{cases} \quad (6.6)$$

From this inclusion we see that functions

$$u^1(t) = \begin{cases} 2t - \frac{1}{2}, & 0 \leq t < \frac{1}{2}, \\ 3t - 1, & \frac{1}{2} \leq t < \frac{7}{9}, \\ t + \frac{5}{9}, & \frac{7}{9} \leq t \leq 1, \end{cases} \quad u^2(t) = \begin{cases} \frac{3}{2}t - \frac{1}{2}, & 0 \leq t < \frac{5}{11}, \\ 2t - \frac{8}{11}, & \frac{5}{11} \leq t < \frac{8}{11}, \\ t, & \frac{8}{11} \leq t \leq 1 \end{cases}$$

are solutions of (6.5). Moreover, each point in the set  $\{(t, u) \mid u^2(t) \leq u \leq u^1(t)\}$  is a doubling bifurcation point for solutions of (6.5). Thus between  $u^2$  and  $u^1$  there is a continuum of chaotically behaving solutions of (6.5). Two other sets of similarly behaving solutions of (6.5) are obtained when  $u^1$  and  $u^2$  above are replaced by

$$u^3(t) = \begin{cases} 2t - 1, & 0 \leq t < \frac{7}{10}, \\ \frac{2}{5}, & \frac{7}{10} \leq t \leq 1, \end{cases} \quad u^4(t) = \begin{cases} \frac{3}{2}t - 1, & 0 \leq t < \frac{2}{3}, \\ 0, & \frac{2}{3} \leq t \leq 1, \end{cases}$$

and

$$u^5(t) = \begin{cases} 2t - \frac{3}{2}, & 0 \leq t < \frac{13}{20}, \\ -\frac{1}{5}, & \frac{13}{20} \leq t \leq 1, \end{cases} \quad u^6(t) = \begin{cases} \frac{3}{2}t - \frac{3}{2}, & 0 \leq t < \frac{13}{21}, \\ -\frac{1}{2}t - \frac{11}{42}, & \frac{13}{21} \leq t \leq 1, \end{cases}$$

respectively. There are no other solutions of (6.5), whence  $u^1$  and  $u^6$  are its extremal solutions.

Because  $H$  and  $x \mapsto [x]$  are right-continuous, a maximal solution  $u^1$  of (6.5) is by Proposition 6.2 a limit of a sequence  $(u_n)_{n=0}^\infty$  defined in (6.4). Calculating its first elements by a computer when  $u_0(t) = 3t$ ,  $t \in J$ , one can infer the exact formula of  $u^1$ . A minimal solution  $u^6$  is obtained in a similar way when e.g.,  $u_0(t) = -2 - t$ ,  $t \in J$ . This confirms that even in

discontinuous cases the iteration chains introduced in Remark 2.1 may be reduced to iteration sequences (cf. [11, p. 9]), and can thus be used to determine extremal solutions of problems in question.

EXAMPLE 6.2. Choose  $J = [0, 1]$ , and define

$$g(t, x) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{h(t, x - m/n)}{2^{|m|/n}}$$

where

$$h(t, x) = \begin{cases} \frac{\sin(x-t)}{x-t}, & x > t, \\ 1, & x = t, \\ \cos \frac{1}{x-t} - 2, & x < t, \end{cases}$$

$$f(t, x, y) = \sum_{n=1}^{\infty} \frac{\arctan([n(x+y-t)])}{n^2}, \quad t \in J, \quad x, y \in \mathbb{R},$$

and

$$C(x, y) = \sum_{n=1}^{\infty} \frac{\arctan([n(x+y)])}{n^2} \quad t \in J \quad x, y \in \mathbb{R}.$$

It is easy to see that  $g, f$  and  $C$  have properties (g0), (g1), (f0), (f1), (C0), and (C1). The nonlinearity  $g$  is discontinuous at each point of the set  $\{(t, x) \mid t \in J, x - t \in \mathbb{Q}\}$ , the set of discontinuity points of  $f$  is  $\{(t, x, y) \mid t \in J, x + y - t \in \mathbb{Q}\}$ , and that of  $C$  is  $\{(x, y) \mid t \in J, x + y \in \mathbb{Q}\}$ . In view of Proposition 6.1 the BVP (6.1), with  $g, f$  and  $C$  as above, has extremal solutions, and they are monotone nondecreasing w.r.t.  $f$  and  $C$ .

Consider next the BVP

$$F(t, u(t), \frac{u'(t)}{q(u(t))} - g(t, u(t))) = 0 \quad \text{for a.a. } t \in J, \quad B(u(t_0), u(t_1)) = 0, \quad (6.8)$$

where the function  $q: \mathbb{R} \rightarrow (0, \infty)$  satisfies condition

(q)  $q$  is measurable and essentially bounded and  $1/q$  is locally essentially bounded.

It follows from [12, Lemma 3.1] that the result of Lemma 3.1 is valid also when the IVP (3.1) is replaced by

$$u'(t) = q(u(t))(g(t, u(t)) + h(t)) \quad \text{for a.a. } t \in J, \quad u(t_o) = x_o. \quad (6.9)$$

Consequently, replacing (3.1) by (6.9) in the definition (3.3) of  $W$ , defining

$$Lu(t) := \frac{u'(t)}{q(u(t))} - g(t, u(t)), \quad u \in W, \quad t \in J,$$

and assuming that the functions  $q$ ,  $g$ ,  $F$  and  $B$  satisfy conditions (q), (g0), (g1), (F0), (F1), (B0), and (B1), the results of Theorem 5.1 and Propositions 5.1 and 5.2 are valid for (6.8). Similarly, the results of Propositions 6.1 and 6.2 are valid for the BVP

$$\begin{cases} u'(t) = q(u(t)) \left[ g(t, u(t)) + f\left(t, u(t), \frac{u'(t)}{q(u(t))} - g(t, u(t))\right) \right] & \text{a.e. in } J, \\ u(t_o) = C(u(t_o), u(t_1)), \end{cases}$$

if condition (q) is added to the hypotheses. These results reveal also a new way to generalize the BVP (1.1), i.e., one can replace  $g(t, u(t))$  in (6.8), (6.9), (3.1), and (3.3) by

$$g(t, u(t)) + f\left(t, u(t), \frac{u'(t)}{q(u(t))} - g(t, u(t))\right).$$

*Remarks.* 6.1. It is easy to see that the multifunction  $\Phi$  given in (6.6) is neither lower- nor upper-semicontinuous, and, in addition, the image  $\Phi(t, u)$  is nonconvex. However, a least smoothness condition for  $\Phi$  in order to apply recent results on multivalued mappings is its lower semi-continuity, cf., e.g., [8, 16]. This shows that other methods described in the Introduction cannot be used to handle problem (6.5) in Example 6.1.

Because of the nature of discontinuities in Example 6.2 it cannot be handled by using results of former theory of implicit differential equations.

We have assumed above that conditions (g1) and (F1) (resp. (g1) and (f1)) hold with the same  $\psi$ . If  $\psi$  replaced by  $\hat{\psi}$  in (g1), we must assume that  $\int_o^\infty dx / \max\{\psi(x), \hat{\psi}(x)\} = \infty$ . This and all the other conditions given for  $\psi : s$  in (g1), (F1), and (f1) hold when  $\psi : s$  are any of the functions:  $\psi_o(x) = ax + b$ ,  $x \geq 0$ , where  $a \geq 0$ ,  $b > 0$ , and

$$\psi_n(x) = (x + 1) \ln(x + e) \cdots \ln_n(x + \exp_n(1)), \quad x \geq 0, \quad n = 1, 2, \dots$$

In view of [12, Lemma 3.1] the results of Theorem 5.1, Propositions 5.1 and 6.1, and Corollary 6.1 are valid also when the hypothesis (g0) is replaced by condition

(g2)  $g$  is sup-measurable, and there is  $p \in L^1(J)$  such that  $x \mapsto g(t, x) + p(t)x$  is monotone nondecreasing for a.a.  $t \in J$ .

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